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Rings with an Additive Rank Function

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INTRODUCTION

In a recent paper Blair and Small [1] study embeddings of Noetherian rings in Artinian rings using a general criterion due to Schofield. This provides conditions on a Sylvester rank under which an embedding is possible and it can be applied to the case which they consider because the reduced rank in the given ring provides an additive Sylvester rank. Since this embedding is into a simple Artinian ring and other embeddings which may be available are not so specific, it is natural to look for an Artinian embedding which covers all cases.

In the present paper we study an extension ring which is appropriate for the case considered by Blair and Small, although at present we can only prove that in general the ring is semi-primary and reduces to the classical quotient ring when that exists. The construction uses a method in torsion theory developed by Johnson, Utumi, Gabriel, and others. The extension ring is a particular subring of the bicommutator of the given ring when it is presented as a ring of endomorphisms on its injective hull. It is worth noting that the full bicommutator ring, usually called the maximal ring of quotients, is unsuited for our purpose, as is clear from the paper of Djabali [2]. In particular, every proper ideal of this ring has a non-zero left annihilator. Moreover, an example is given in Schelter and Small [8] of an Artinian ring whose maximal ring of quotients is not Artinian. On the positive side, however, we show that it retains the property of being semi-primary.

Although a short introduction has been given here, the reader is referred to Lambek [5] and Stenström [10] for details and proofs of basic results on rings of endomorphisms of the injective hull.

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1. Preliminaries

Let R be a ring. A linear topology is introduced in R by the specification of a set $\mathscr{F}$ of right ideals such that

(i) if $F \in \mathscr{F}$ and $I \supset F$ is a right ideal, then $I \in \mathscr{F}$;

(ii) if $F_1, F_2 \in \mathscr{F}$ then $F_1 \cap F_2 \in \mathscr{F}$;

(iii) if $F \in \mathscr{F}$ and $a \in R$ then $(x \in R; ax \in F) \in \mathscr{F}$;

(iv) if I is a right ideal and there exists $F \in \mathscr{F}$ such that $(x \in R; ax \in I) \in \mathscr{F}$ for every $a \in F$, then $I \in \mathscr{F}$.

We use the term *Gabriel filter* to denote a set with these properties. It can be verified that (i) and (ii) are consequent on (iii) and (iv).

Let M be a right R-module. For a submodule N define the $\mathscr{F}$*-closure* of N in M by

$$\operatorname{cl} N = \operatorname{cl}(N, \mathscr{F}) = (m \in M; mF \subset N \text{ for some } F \in \mathscr{F}).$$

Condition (iv) implies that $\operatorname{cl} \operatorname{cl}(N) = \operatorname{cl}(N)$, which means that $\operatorname{cl}(N)$ is an $\mathscr{F}$-closed submodule of M. For further details on Gabriel filters see Stenström [10].

Our aim is to study the structure of a ring R with a Gabriel filter $\mathscr{F}$ of right ideals which has the following conditions:

(A1) $\ker \mathscr{F} = 0$;

(A2) the lattice of $\mathscr{F}$-closed right ideals of R is Artinian.

The first condition asserts that if $F \in \mathscr{F}$ and $x \in R$ then $xF = 0$ implies that $x = 0$; in other words, $\operatorname{cl} 0 = 0$. The second asserts that the lengths of chains of $\mathscr{F}$-closed right ideals of R have a finite upper bound which is the $\mathscr{F}$*-rank* of R and is denoted by $\rho(R, \mathscr{F})$, or shortly $\rho(R)$.

The lattice operations in (A2) are

$$\operatorname{cl} I \cap \operatorname{cl} J = \operatorname{cl}(I \cap J) \qquad \text{and} \qquad \operatorname{cl} I \cup \operatorname{cl} J = \operatorname{cl}(\operatorname{cl} I + \operatorname{cl} J).$$

It should be noted that for closed right ideals $I \supset J$, the lengths of saturated chains of closed right ideals between I and J are equal. When $J = 0$ this number is the $\mathscr{F}$-rank of I, denoted by $\rho(I)$ or $\rho(I, \mathscr{F})$. The common length of saturated chains between I and J is $\rho(I) - \rho(J)$.

An *additive rank function* assigns to each finitely generated right R-module M a non-negative integer $\rho(M)$ such that, for every short exact sequence of these modules $0 \to N \to M \to L \to 0$, $\rho(M) = \rho(N) + \rho(L)$. The function is *non-singular* if $\rho(I) = 0$, for a right ideal I, implies that $I = 0$.

PROPOSITION 1.1. *Let ρ be an additive rank function for a right Noetherian ring R and set $\mathcal{F}=(F\in R_R;\ \rho(R/F)=0)$. Then $\mathcal{F}$ is a Gabriel filter and the lattice of $\mathcal{F}$-closed right ideals is Artinian.*

The proof is left to the reader.

In Krause [4] it is shown that for a right Noetherian ring, the property (A2) also holds for the lattice of $\mathcal{F}$-closed submodules of a finitely generated module.

Let R be a right Noetherian ring with nilpotent radical N and let $\mathscr{C}(N)$ be the set of elements of R which are regular modulo N. Define

$$K=(a\in R;\ \text{for each } r\in R \text{ there exists } c\in\mathscr{C}(N) \text{ with } arc=0).$$

Then K is an ideal; used in Blair and Small [1].

PROPOSITION 1.2. *Let R be a right Noetherian ring such that the ideal K is zero. Then R has a Gabriel filter which satisfies* (A1) *and* (A2).

Proof. Let $\rho(\cdot)$ be the reduced rank function on R_R. Define $\mathcal{F}=(F\in R_R;\ \rho(R/F)=0)$. In fact $F\in\mathcal{F}$ provided that, for each $a\in R$, there exists $c\in\mathscr{C}(N)$ such that $ac\in F$. For right ideals $I\supset J$ we have $\rho(I)=\rho(\mathrm{cl}\ I)$ and $\rho(I)=\rho(J)$ if and only if $\mathrm{cl}\ I=\mathrm{cl}\ J$. Lengths of chains of $\mathcal{F}$-closed right ideals are bounded above by $\rho(R)$ and the maximal length of such a chain from $\mathrm{cl}\ I$ to $\mathrm{cl}\ J$ is $\rho(I)-\rho(J)$. Evidently $K=0$ is equivalent to asserting that $\ker\mathcal{F}=0$.

EXAMPLES. 1. A right Noetherian ring with zero right singular ideal satisfies 1.2.

2. A left and right Noetherian ring such that $\mathrm{Ass}(R_R)$ has only minimal primes satisfies 1.2. When the ring is also commutative then it has an Artinian quotient ring, but in general this will not hold even in Example 1.

2. ENDOMORPHISMS OF THE INJECTIVE HULL OF R_R

Let V be the injective hull of R as a right module and set $H=\mathrm{End}\ V_R$. The following result is well-known; see Lambek [5].

PROPOSITION 2.1. (i) *$V=H1$, where 1 is the unity element of R;*

(ii) *$J=(h\in H;$ ker h is an essential submodule of $V_R)$ is the Jacobson radical of H;*

(iii) *H/J is a von Neumann regular ring.*

It should also be noted that the map $h\to h1$ is an epimorphism $H\to V$ as left

H-modules and the kernel $J_0=(h \in H; h1=0)$ is a left ideal of H. The proposition can be sharpened when R has a Gabriel filter $\mathscr{F}$ which satisfies the basic assumptions (A1) *and* (A2). *Let $U_1 + \cdots U_n$ be a direct sum of right ideals; then* $\mathrm{cl}\, U_1 + \cdots + \mathrm{cl}\, U_n$ *is also a direct sum which by assumption* (2) *has length less than or equal to $\rho(R, \mathscr{F})$. It follows that V is the direct sum of at most $\rho(R, \mathscr{F})$ indecomposable injectives and that H/J is a semi-simple Artinian ring. Note also that R has the acc condition for right annihilators, because any right annihilator is $\mathscr{F}$-closed.*

THEOREM 2.2. *Let R be a ring with a Gabriel filter $\mathscr{F}$ of right ideals which satisfies the basic assumptions* (A1) *and* (A2). *Using the notation of Proposition* (2.1),

(i) *the radical J of H is nilpotent with index $\leqslant \rho(R, \mathscr{F})$;*

(ii) *H is a semi-primary ring;*

(iii) *${}_H V$ is an Artinian module.*

Proof. Let $x_1, ..., x_m \in J$, where $m=\rho(R, \mathscr{F})$. Either $x_1 R=0$ and then $x_1 \in J_0$ or $\ker x_2 \cap x_1 R \cap R \neq 0$. The latter implies that $R \cap \ker(x_2 x_1) \supsetneqq R \cap \ker(x_1)$, because these are $\mathscr{F}$-closed right ideals of R. Continuing, for $k=1, 2, ...$ we have either $x_k x_{k-1} \cdots x_1 \in J_0$ or the series

$$R \cap \ker x_1 \subsetneqq R \cap \ker(x_2 x_1) \subsetneqq \cdots \subsetneqq R \cap \ker(x_k \cdots x_1) \subsetneqq \cdots$$

which has strictly increasing rank. This series has $\leqslant m$ terms and thus each case leads to $x_m \cdots x_1 \in J_0$. It follows that $J^m \subset J_0$ and $J^m V = J^m H1 = J^m 1 = 0$, so that $J^m = 0$.

To obtain (iii) it is enough to show that H/J_0 is a left Noetherian H-module since $H/J_0 \approx V$ as H-modules. Let L be a left ideal of H such that $L \supset J_0$. Denoting $\ker L = (\bigcap \ker h; h \in L)$, there exists a finite set $x_1, ..., x_s \in L$ such that $R \cap \ker L = R \cap \ker x_1 \cap \cdots \cap \ker x_s$. Choose $x \in L$ and consider the map

$$(x_1, ..., x_s)\, y = (x_1 y, ..., x_s y) \to xy \qquad (y \in R)$$

of $L \oplus \cdots \oplus L$ with s terms into L, which extends to an R-hom $V^s \to V$. The latter is an element of $H \oplus \cdots \oplus H$ with s terms. Hence there exist $h_1, ..., h_s \in H$ such that

$$(h_1 x_1 + \cdots + h_s x_s)\, y = xy \qquad (y \in R).$$

Thus $Hx + J_0 \subset Hx_1 + \cdots + Hx_s + J_0$. Since $x \in L$ is arbitrary it follows that $L = Hx_1 + \cdots + Hx_s + J_0$ and (iii) follows at once.

COROLLARY. *Let R e a right Noetherian ring with zero right singular ideal. Then H is a semi-simple Artinian ring and R embeds naturally in H^{op}.*

Proof. The relevant $\mathscr{F}$ is the set of essential right ideals of R and $J = J_0 = 0$.

3. THE BICOMMUTATOR OF R WITH RESPECT TO V

If R is considered as a ring of endomorphisms on its injective hull V, the latter becomes a bimodule ${}_H V_R$. Let $T = \mathrm{End}({}_H V)$ be the bicommutator of R. It is convenient to place the maps from T on the right of V, thus

$$(hv)t = h(vt); \qquad h \in H, v \in V, t \in T,$$

and to regard R as a subring of T. The ring T is referred to in the literature variously as the *maximal, complete*, or Utumi ring of quotients. A full account of the general properties of T is to be found in Lambek [5] and in Passman [7]. We need the following result from Lambek [5].

PROPOSITION 3.1. (i) *The map $t \to 1t$ for $t \in T$ is a monomorphism of right R-modules;*

(ii) *the ring T is anti-isomorphic to the idealiser eigen ring K/J_0, where $K = (h \in H; J_0 h \subset J_0)$;*

(iii) *if the map $t \to 1t$ is surjective and the basic assumptions hold then T is anti-isomorphic to H and they are semi-simple Artinian rings.*

In the general case a Gabriel filter can be defined on T. A submodule D of T_R is said to be *dense* if for $h \in H$, $hD = 0$ implies that $h \in J_0$. The following result is proved in Lambek [5].

PROPOSITION 3.2. *Let D and G be submodules of T_R with D dense, then*

$$\mathrm{Hom}_R(D, G) \approx G:D = (t \in T; tD \subset G).$$

The isomorphism is $d \to td$ for $d \in D$.

One consequence of this result is that T is isomorphic to the direct limit $\lim_{\rightarrow}(\mathscr{D}, R)$ taken over the Gabriel filter $\mathscr{D}$ of dense right ideals D. This ring is the localised ring of R under the Gabriel filter. It has a subring arising from any given Gabriel subfilter of $\mathscr{D}$. Let R be a right Noetherian ring such that the ideal K of Proposition 1.2 is zero. The filter $\mathscr{F}$ defined in Proposition 1.1 is a Gabriel subfilter of $\mathscr{D}$. For if $hF = 0$, where $h \in H$ and $F \in \mathscr{F}$, then $(h1)F = 0$. Now the kernel of $\mathscr{F}$ in V is an R-submodule of V and its intersection with R_R is ker $\mathscr{F}$,which is zero. Hence $h1 = 0$, which shows that F is dense. We denote $S = \mathrm{Lim}_{\to}(F, R)$, where $F \in \mathscr{F}$. The structure of rings S and T can be partially analysed with the help of Theorem 1.

PROPOSITION 3.3. *Let $V = V_0 \supset V_1 \supset \cdots \supset V_k = 0$ be a composition series of (H, S)-bimodules and let $Q_1, ..., Q_k$ be the right annihilators of the factors V_i/V_{i+1} for $i = 0, ..., k-1$. Then $Q_1, ..., Q_k$ are prime ideals of S, including the minimal primes, and the prime radical of S is nilpotent.*

Proof. Evident from Theorem 2.2 and the fact that $Q_1 \cdots Q_k = 0$.

It is worth noting that a similar result holds for any ring Y lying between R and T, so that, in particular, the prime radical of Y is nilpotent. Note that if R has a classical quotient ring then this is Artinian and isomorphic to S. In this case the prime ideals of S in 3.3 are all minimal. Conversely, if the latter holds, then R has an Artinian quotient ring. It follows that in general some of the Q_i are not minimal and this can be recognized from the property $Q_i \in \mathscr{F}$.

THEOREM 3.4. *Let R be a right Noetherian ring which has a non-singular additive rank function ρ and let S be the ring obtained by localising at the Gabriel filter $\mathscr{F}$ defined by $\mathscr{F} = (F \in R_R;\, \rho(R) = \rho(F))$. Then S is a semi-primary ring and its right socle is an essential right ideal, which is Artinian.*

Proof. Let I be a non-nilpotent right ideal of S. Now S has finite uniform dimension on right ideals which it inherits from R. Let $A_1 \subset A_2 \subset \cdots$ be an ascending chain of right annihilators in S; then the chain $(A_i \cap R)$, $i = 1, 2, ...$, consists of F-closed right ideals of R, and hence must become stationary. It follows that the chain A_i, $i = 1, 2, ...$ also becomes stationary. So S has ascending chain condition on right annhilators. By a theorem of Lanski [6] the right ideal I has a non-nilpotent element s. For some $n > 0$ we have

$$(x \in R;\, s^n x = 0) = \mathrm{ann}(s^n) = \mathrm{ann}(s^{2n}) \text{ and } s^n R \cap \mathrm{ann}(s^n) = 0.$$

The exact sequence $0 \to \mathrm{ann}(s^n) \to R \to s^n R \to 0$ shows that $\rho(R) = \rho(\mathrm{ann}\, s^n) + \rho(s^n R)$. Suppose that $F \in \mathscr{F}$ such that $s^n F \subset R$; then it follows that $s^n F \oplus \mathrm{ann}(s^n) \in \mathscr{F}$. Define $u, u' \in S$ by specifying them as R-homs on elements of $\mathscr{F}$. Thus

$$u(s^{2n}x + y) = s^n x + y \qquad \text{and} \qquad u'(s^n x + y) = s^{2n}x + y.$$

Then $uu' = u'u = 1$, where 1 is the unity element of S. Moreover

$$(s^n u s^n u - s^n u)(s^{2n}F + \mathrm{ann}(s^n)) = 0,$$

so that $e = s^n u$ is an idempotent element of I. Then $I = eI + (1 - e)I$ and either $(1 - e)I$ is a nilpotent right ideal or it has an idempotent element f. In the latter case, if we set $g = e + f(1 - e)$, then $gR = eR \oplus fR$. Continuing the argument, eventually we obtain an idempotent $g \in I$ such that $I = gR \oplus (1 - g)I$ and $(1 - g)I$ is a nilpotent right ideal of S. Let N be the nilpotent radical of the ring S. Then by the argument thus far, the ring S/N is a semi-simple artin ring and S is a semi-primary ring.

Finally, the right socle of S is the left annihilator of N; it is a direct sum of a finite number of minimal right ideals since S has finite uniform dimension.

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References

1. W. D. Blair and L. W. Small, Embeddings in artinian rings and Sylvester rank functions, *Israel J. Math.* **58** (1987), 10–18.
2. M. K. Djabali, Étude de la condition artinienne, *Bull. Sci. Math. (2)* **98** (1974), 145–161.
3. G. Krause, Additive rank functions in Noetherian rings, *J. Algebra*, in press.
4. G. Krause, Additive rank functions and Gabriel filters in right Noetherian rings, *J. Algebra*, inpress.
5. J. Lambek, "Lectures on Rings and Modules," Ginn (Blaisdell), Boston, 1966.
6. C. Lanski, Nil subrings of Goldie rings are nilpotent, *Canad. J. Math.* **21** (1969), 904–907.
7. D. S. Passman, "The Algebraic Structure of Group Rings," Wiley–Interscience, New York, 1977.
8. W. Schelter and L. W. Small, Some pathological rings of quotients, *J. London Math. Soc.* (2) **14** (1976), 200–202.
9. L. W. Small, The embedding problem for Noetherian rings, *Bull. Amer. Math. Soc.* **75** (1969), 147–148.
10. B. Stenström, "Rings of Quotients," Springer-Verlag, Berlin/New York, 1975.
11. R. B. Warfield, Jr., Quotient rings and localisation for Noetherian rings, *J. Algebra* **72** (1981), 166–182.